# Scaling in a general class of critical random Boolean networks

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We derive analytically the scaling behavior in the thermodynamic limit of the number of nonfrozen and relevant nodes in the most general class of critical Kauffman networks for any number of inputs per node, and for any choice of the probability distribution for the Boolean functions. By defining and analyzing a stochastic process that determines the frozen core we can prove that the mean number of nonfrozen nodes in any critical network with more than one input per node scales with the network size $N$ as $N^{2/3}$, with only $N^{1/3}$ nonfrozen nodes having two nonfrozen inputs and the number of nonfrozen nodes with more than two inputs being finite in the thermodynamic limit. Using these results we can conclude that the mean number of relevant nodes increases for large $N$ as $N^{1/3}$, with only a finite number of relevant nodes having two relevant inputs, and a vanishing fraction of nodes having more than three of them. It follows that all relevant components apart from a finite number are simple loops, and that the mean number and length of attractors increases faster than any power law with network size.

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## I. INTRODUCTION

Random Boolean networks are often used as generic models for the dynamics of complex systems of interacting entities, such as social and economic networks, neural networks, and gene or protein interaction networks [1]. The simplest and most widely studied of these models was introduced in 1969 by Kauffman [2] as a model for gene regulation. The system consists of $N$ nodes, each of which receives input from $K$ randomly chosen other nodes. The network is updated synchronously, the state of a node at time step $t$ being a Boolean function of the states of the $K$ input nodes at the previous time step, $t-1$. The Boolean updating functions are randomly assigned to every node in the network, and together with the connectivity pattern they define the realization of the network. For any initial condition, the network eventually settles on a periodic attractor.

Of special interest are *critical* networks, which lie at the boundary between a frozen phase and a chaotic phase [3,4]. In the frozen phase, a perturbation at one node propagates during one time step on an average to less than one node, and the attractor lengths remain finite in the limit $N\to\infty$. In the chaotic phase, the difference between two almost identical states increases exponentially fast because a perturbation propagates on an average to more than one node during one time step [5]. Whether a network is frozen, chaotic, or critical depends on the connectivity $K$ as well as on the weights of the different Boolean functions. If these weights are chosen appropriately, critical networks can be created for any value of $K$.

The nodes of a critical network can be classified according to their dynamics on an attractor. First, there are nodes that are frozen on the same value on every attractor. Such nodes give a constant input to other nodes and are otherwise irrelevant. They form the *frozen core* of the network. Second, there are nodes whose outputs go only to irrelevant nodes. Though they may fluctuate, they are also classified as irrelevant since they act only as slaves to the nodes determining the attractor period. Third, the *relevant nodes* are the nodes whose state is not constant and that control at least one relevant node. These nodes determine completely the number and period of attractors. If only these nodes and the links between them are considered, they form loops with possibly additional links and chains of relevant nodes within and between loops. The recognition of the relevant elements as the only elements influencing the asymptotic dynamics was an important step in understanding the attractors of Kauffman networks. The behavior of the frozen core was first studied by Flyvbjerg [6]. Then, in an analytical study of $K=1$ networks Flyvbjerg and Kjaer [7] introduced the concept of relevant elements. This concept was generalized to general critical networks by Bastolla and Parisi [8,9]. They gained insight into the properties of the attractors of the critical networks by using numerical experiments based on the modular structure of the relevant elements. Finally, Socolar and Kauffman [10] found numerically that for critical $K=2$ networks the mean number of nonfrozen nodes scales as $N_{nf}\sim N^{2/3}$, and the mean number of relevant nodes scales as $N_{rel}\sim N^{1/3}$. The same result is hidden in the analytical work on attractor numbers by Samuelsson and Troein [11], as was shown in [12]. An explicit analytical derivation of these and other scaling laws was given in [13]. For $K=1$, these power laws are $N_{nf}\sim N$ and $N_{rel}\sim N^{1/2}$, since there is no frozen core in a $K=1$ critical network.

In this work, we will derive the scaling behavior of the number of nonfrozen and of relevant nodes in critical Kauffman networks with $K\geqslant 3$. Since the scaling behavior is different for $K=1$ and 2, one could expect that the exponents are generally $K$-dependent. However, we will show that the exponents $2/3$ and $1/3$ found for $K=2$ are valid also for larger $K$ and for all possible probability distributions of the Boolean functions, as long as the network is critical. We also obtain results for the number of nonfrozen nodes with two and more nonfrozen inputs, and for the number of relevant nodes with two and more relevant inputs.

The outline of this paper is the following. In the next section, we introduce a stochastic process that yields the frozen core in $K=3$ networks. The mean-field theory for this process is presented in Sec. III, and an improved treatment

 

including fluctuations is presented in Sec. IV, yielding the scaling behavior of the number of nonfrozen nodes in critical networks. The next three sections are devoted to special points in parameter space, where the stochastic process does not generate all of the frozen core. In Secs. V and VI those points are considered, where the stochastic process gives a smaller frozen core, and it is shown that "self-freezing loops" generate the rest of the frozen core. In Sec. VII, we consider points in parameter space, where the stochastic process does not generate any frozen nodes, and where self-freezing loops are responsible for all of the frozen core. Finally, in Secs. VIII and IX we evaluate the case $K \geqslant 4$ and the scaling behavior of the relevant nodes and attractor properties. Section X discusses the implications of our results.

## II. A STOCHASTIC PROCESS THAT LEADS TO THE FROZEN CORE

From now on, we set $K=3$ and derive explicitly the scaling behavior of the nonfrozen nodes. The generalization to larger $K$ and the scaling behavior of the relevant nodes will be discussed later. The first step of the calculation, which is performed in this section, consists in defining a stochastic process that determines the frozen core. This process is inspired by the one used in [13] for $K=2$, however, it needed to be modified before it could be generalized to larger $K$. The treatment presented in the following is based on the existence of nodes with constant functions (functions in which the output is fixed irrespectively of the input) and it therefore applies to all critical models that have a nonzero fraction of constant functions. Networks with no constant functions, and in particular networks with only canalyzing functions will be discussed separately.

Flyvbjerg [6] was the first one to use a dynamical process that starts from the nodes with constant update functions and determines iteratively the frozen core. Performing a mean-field calculation for this process, he could identify the critical point. We define in the following a process that goes beyond mean-field theory and gives exact results for the frozen core. We consider the ensemble of all networks of size $N$ with a fixed number of nodes with constant update functions. All nodes with a constant update function are certainly part of the frozen core. We construct the frozen core by determining stepwise all those nodes that become frozen due to the influence of a frozen node. In the language of [10], this process determines the "clamped" nodes.

In a $K=3$ network, each node has three inputs, and there are consequently $2^{2^3}=256$ possible Boolean functions. In order to specify a model, one has to specify the probabilities for a node to choose each of these functions. Instead of performing the calculation in terms of all these parameters, it turns out that three parameters are sufficient. For the $K=2$ networks, we introduced three parameters corresponding to the occurrence of three types of Boolean functions. For larger $K$, there are more types of Boolean functions, and we use therefore a different set of parameters. The first parameter is $\beta$, which is the proportion of nonfrozen nodes in the network. $1-\beta$ is therefore the proportion of nodes with a constant update function. We require $\beta<1$ for the calculation performed in this and the following section. The case $\beta=1$ will be discussed further below. The second parameter is $\omega_2$, which is the probability that a randomly chosen node that does not have a constant update function will become a frozen node when one of its three inputs is connected to a frozen node. If one input of a node is fixed at some value, the node has effectively two inputs left. We now consider those nodes that have not become frozen by fixing one input, i.e., we are considering the proportion $1-\omega_2$ of all nonfrozen nodes. The parameter $\omega_1$ is then the probability that such a node becomes frozen when one of the remaining two inputs is connected to a frozen node. This probability can again be expressed in terms of the probabilities of the different possible update functions. Thus all the networks with the same parameters $\omega_2$, $\omega_1$, and $\beta$ will be treated as of the same type. As we will see below, the properties we are interested in will be the same not only for the functions that belong to the same type of network (i.e., that have the same parameters but possibly different Boolean functions) but also for the different types as long as their parameters are such that the network satisfies the criticality condition (3) derived below. This means that we can have critical networks with all possible choices of Boolean functions and that they will all be characterized by the same exponents as a consequence of being critical.

Now, let us define the stochastic process that determines the frozen core. For this purpose, we differentiate four types of nodes, the numbers of which will change during the process, and we place these nodes in four different "containers." Initially, all nodes with constant functions are placed in a container labeled $\mathcal{F}$, and the remaining nodes in a container labeled $\mathcal{N}_3$. In this container are all those nodes for which we do not yet know if they are connected to a frozen node. The other two containers, labeled $\mathcal{N}_2$ and $\mathcal{N}_1$, are initially empty. They will contain nodes with one and two frozen inputs that are themselves not (yet) frozen. Since the number of nodes in the different containers is going to change during our stochastic process, we denote the initial values of numbers of nodes in the containers as $N_f^{ini}$, $N_2^{ini}=N_1^{ini}=0$, and $N_3^{ini}$, and the total number of nodes as $N^{ini}$ (this is the actual number of nodes in the network). The contents of the containers will change with time. The "time" we are defining here is not the real time for the dynamics of the system. Instead, it is the time scale for a stochastic process that we use to determine the frozen core. During one time step, we choose one node from the container $\mathcal{F}$ and determine the influence of this node on the nodes connected to it. After determining its influence we will remove it from the system, and the number of nodes $N$ in the system is reduced by 1. Now, for each nonfrozen node in container $\mathcal{N}_3$ we ask whether it receives input from the chosen frozen node. If this is the case it freezes with probability $\omega_2$ due to the influence of this node and moves to container $\mathcal{F}$. With probability $1-\omega_2$ it does not become frozen and moves to container $\mathcal{N}_2$. In one time step, we therefore move each node of container $\mathcal{N}_3$ with probability $3\omega_2/N$ to the container $\mathcal{F}$, and with probability $3(1-\omega_2)/N$ to the container $\mathcal{N}_2$. Similarly, a node from the container $\mathcal{N}_2$ receives input from the chosen frozen node with probability $2/N$, and it will then become frozen with probability $\omega_1$ and will be placed in the container $\mathcal{F}$. If it does not

freeze, we place it in container $\mathcal{N}_1$, where we find all those nodes that have two inputs from frozen nodes and are not frozen. When nodes from this container choose a frozen node as an input, they automatically become frozen. During this process, the probabilities $\omega_2$ and $\omega_1$ will not change since the nodes from containers $\mathcal{N}_3$ and $\mathcal{N}_2$, for which we are in every time step determining whether they are going to freeze, are chosen at random, and moving them from the containers will not change probability distribution of the functions of the nodes left in the containers. In the next time step, we choose another frozen node from container $\mathcal{F}$ and determine its effect on the other nodes. Some nodes move again to a different container, and the chosen frozen node is removed from the system. We repeat this procedure until we cannot continue because either container $\mathcal{F}$ is empty, or because all the other containers are empty. If container $\mathcal{F}$ becomes empty, we are left with the nonfrozen nodes. We shall see below that most of the remaining nodes are in container $\mathcal{N}_1$, with the proportion of nodes left in containers $\mathcal{N}_2$ and $\mathcal{N}_3$ vanishing in the limit $N^{ini} \to \infty$. If all containers apart from container $\mathcal{F}$ are empty at the end, the entire network becomes frozen. This means that the dynamics of the network goes to the same fixed point for all initial conditions.

## III. MEAN-FIELD APPROXIMATION AND THE CRITICALITY CONDITION

Let us first describe this process by deterministic equations that neglect fluctuations around the average change of the number of nodes in the different containers. As long as all containers contain large numbers of nodes, these fluctuations are negligible, and the deterministic description is appropriate. The average change of the node numbers in the containers during one time step is

$$\Delta N_3 = -\frac{3N_3}{N},$$

$$\Delta N_2 = -\frac{2N_2}{N} + (1-\omega_2)\frac{3N_3}{N},$$

$$\Delta N_1 = -\frac{N_1}{N} + (1-\omega_1)\frac{2N_2}{N},$$

$$\Delta N_f = -1 + \frac{N_1}{N} + \omega_1\frac{2N_2}{N} + \omega_2\frac{3N_3}{N},$$

$$\Delta N = -1. \tag{1}$$

The total number of nodes in the containers, $N$, can be used instead of the time variable, since it decreases by one during each step. The equation for $N_3$ can then be solved by going from a difference equation to a differential equation,

$$\frac{\Delta N_3}{\Delta N} \simeq \frac{dN_3}{dN} = -\frac{3N_3}{N},$$

which has the solution

$$N_3 = N^3\frac{N_3^{ini}}{(N^{ini})^3} = \frac{\beta}{(N^{ini})^2}N^3,$$

where $\beta = \frac{N_3^{ini}}{N^{ini}}$. Similarly, we find

$$N_2 = 3(1-\omega_2)\frac{\beta}{N^{ini}}N^2 - 3(1-\omega_2)\frac{\beta}{(N^{ini})^2}N^3,$$

$$N_1 = 3(1-\omega_1)(1-\omega_2)\beta N - 6(1-\omega_1)(1-\omega_2)\frac{\beta}{N^{ini}}N^2 + 3(1-\omega_1)(1-\omega_2)\frac{\beta}{(N^{ini})^2}N^3,$$

$$N_f = [1-3(1-\omega_1)(1-\omega_2)\beta]N + 3(1-2\omega_1)(1-\omega_2)\frac{\beta}{N^{ini}}N^2 + [3\omega_1(1-\omega_2)-1]\frac{\beta}{(N^{ini})^2}N^3. \tag{2}$$

When $1-3(1-\omega_1)(1-\omega_2)\beta < 0$, the equation $N_f = 0$, which represents the stopping condition for the process, has a solution for a nonzero value $N$. This solution shows that the number of nonfrozen nodes in each container is proportional to $N^{ini}$. This means that on an average a nonfrozen node has more than one nonfrozen input. A perturbation at one node propagates during one time step on an average to more than one node and we are obviously in the chaotic phase.

For $1-3(1-\omega_1)(1-\omega_2)\beta \geqslant 0$ the equation $N_f = 0$ does not have a nonzero solution for $N \in [0, N^{ini}]$. In this case, we will stop the process when $N_f$ drops below 1. We are in the frozen phase, or we have a critical system.

In the case $1-3(1-\omega_1)(1-\omega_2)\beta > 0$, the values $N_3$ and $N_2$ will sink below 1 when $N$ becomes of the order $\sqrt{N^{ini}}$, and the higher-order terms contributing to $N_f$ and $N_1$ can be neglected compared to the first one. For smaller $N$, only frozen nodes and nodes with one input are left. When $N_f$ falls below 1, there remain only a constant number of the nodes of type $\mathcal{N}_1$,

$$N_1 \simeq \frac{3(1-\omega_1)(1-\omega_2)\beta}{1-3(1-\omega_1)(1-\omega_2)\beta}.$$

The network is essentially frozen, with only a finite number of nonfrozen nodes in the limit $N^{ini} \to \infty$. If we now choose the inputs for these nodes, we obtain simple loops with trees rooted in the loops. This property of the frozen phase was also found in [10].

When parameters of the networks are such that

$$1-3(1-\omega_1)(1-\omega_2)\beta = 0 \tag{3}$$

is fulfilled, we are at the boundary between frozen and chaotic phase in the parameter space. Thus the network is critical. Since the stochastic process stops at $N_f = 1$, we have

$$1 = \frac{(1-2\omega_1)}{(1-\omega_1)}\frac{(N^{end})^2}{N^{ini}} + \left(\frac{\omega_1}{(1-\omega_1)} - \beta\right)\frac{(N^{end})^3}{(N^{ini})^2}.$$

In the limit $N^{ini} \to \infty$ the first term is dominant and the number of nonfrozen nodes would scale with the square root of

the network size if the deterministic approximation to the stochastic process was exact. We shall see below that including fluctuations changes the exponent from 1/2 to 2/3. The final number of $\mathcal{N}_2$-nodes for the deterministic process for the critical networks is independent of network size, and the final number of $\mathcal{N}_3$-nodes is $\sim(N^{ini})^{-1/2}$ and vanishes for $N^{ini}\to\infty$. We shall see below that the fluctuations change these two results to $N_2\sim(N^{ini})^{1/3}$ and $N_3\sim$const.

The deterministic description of our process gives the wrong scaling of the number of nonfrozen nodes in the case of critical networks, but a correct criticality condition (3). We are interested in the dynamical behavior of the networks in the critical phase and we will from now on study only networks with the parameters such that the criticality condition $1-3(1-\omega_1)(1-\omega_2)\beta=0$ is fulfilled.

Before we proceed by introducing the noise into the deterministic equations, there is one more piece of information we can extract from the deterministic description of the critical process that is going to help us later in determining the noise term. Introducing $n=N/N^{ini}$ and $n_j=N_j/N^{ini}$ for $j=f,1,2,3$, Eqs. (2) simplify to (using the criticality condition)

$$n_3=\beta n^3,$$

$$n_2=\frac{1}{1-\omega_1}(n^2-n^3),$$

$$n_1=n-2n^2+n^3,$$

$$n_f=\frac{1-2\omega_1}{1-\omega_1}n^2+\left(\frac{\omega_1}{1-\omega_1}-\beta\right)n^3. \tag{4}$$

This means that our stochastic process remains invariant (in the deterministic approximation) when the initial number of nodes in the containers and the time unit are all multiplied by the same factor. For small $n$, the majority of nodes are in container $\mathcal{N}_1$, since $n_1=n-O(n^2)$. Now, if we choose a sufficiently large $N^{ini}$, $n$ reaches any given small value while $N_f\sim n^2N^{ini}$ is still large enough for a deterministic description. We can therefore assume that for sufficiently large networks $N_f/N\sim n$ becomes small before the effect of the noise becomes important. This assumption will simplify our calculations below.

## IV. THE EFFECT OF FLUCTUATIONS

The number of nodes in container $\mathcal{N}_j$, $j=1,2,3$, that choose a given frozen node as an input is Poisson distributed with a mean $jN_j/N$ and a variance $jN_j/N$. We now assume that $n$ is small at the moment where the noise becomes important, i.e., that the variance of the three noise terms is $N_1/N=n_1/n=1-2n+n^2=1-O(n)$ and $2N_2/N=2n_2/n=\frac{2}{1-\omega_1}(n-n^2)=O(n)$ and $3N_3/N=3\beta n^2=O(n^2)$. All three noise terms occur in the equation for $N_f$, and since the first term dominates for small $n$, we consider only this term in the equation for $N_f$. In the equations for $N_1$ and $N_2$, the noise term is much smaller than the number of nodes in these containers and can therefore be dropped.

The effect of the noise on the final value of $N_3$ can be obtained by the following consideration: as we will see below, the mean final value of $N_3$ will be a constant, which is independent of $N^{ini}$. This means that each node that is initially in the container $\mathcal{N}_3$ has a probability of the order $1/N^{ini}$ of never choosing a frozen input during the stochastic process, and this probability is independent for each node. From this follows that the final number $N_3$ is Poisson distributed with a variance that is identical to the mean. This variance is finite in the limit $N^{ini}\to\infty$ and it does not affect the final value of $N_2$ or $N_1$. Since we have obtained the variance of the final value of $N_3$ by this simple argument, we will not explicitly consider the noise term in the equation for $N_3$.

We therefore obtain the stochastic version of Eqs. (1), where we need to retain only the noise term in the equation for $N_f$:

$$\Delta N_3=-\frac{3N_3}{N},$$

$$\Delta N_2=-\frac{2N_2}{N}+\frac{1}{\beta(1-\omega_1)}\frac{N_3}{N},$$

$$\Delta N_f=-1+\frac{N_1}{N}+2\omega_1\frac{N_2}{N}+\left(3-\frac{1}{\beta(1-\omega_1)}\right)\frac{N_3}{N}-\xi,$$

$$\Delta N=-1. \tag{5}$$

The random variable $\xi$ has zero mean and unit variance. As long as the $n_j$ change little during one time step, we can summarize a large number $T$ of time steps into one effective time step, with the noise becoming Gaussian distributed with zero mean and variance $T$. Exactly the same process would result if we summarized $T$ time steps of a process with Gaussian noise of unit variance. For this reason, we can choose the random variable $\xi$ to be Gaussian distributed with unit variance.

Compared to the deterministic case, the equations for $N_3$ and $N_2$ are unchanged. Inserting the solution for $N_3$ and $N_2$ into the equation for $N_f$, we obtain

$$\frac{dN_f}{dN}=\frac{N_f}{N}+\frac{1-2\omega_1}{1-\omega_1}\frac{N}{N^{ini}}+2\left(\frac{\omega_1}{1-\omega_1}-\beta\right)\left(\frac{N}{N^{ini}}\right)^2+\xi \tag{6}$$

with the step size $dN=1$ and $\langle\xi^2\rangle=1$. [In the continuum limit $dN\to0$ the noise correlation becomes $\langle\xi(N)\xi(N')\rangle=\delta(N-N')$.] This is a Langevin equation, and the corresponding Fokker-Planck equation is

$$-\frac{\partial P}{\partial N}=\frac{\partial}{\partial N_f}\left[\frac{N_f}{N}+\frac{1-2\omega_1}{1-\omega_1}\frac{N}{N^{ini}}+2\left(\frac{\omega_1}{1-\omega_1}-\beta\right)\left(\frac{N}{N^{ini}}\right)^2\right]P+\frac{1}{2}\frac{\partial^2P}{\partial N_f^2}. \tag{7}$$

Since we are investigating networks in the thermodynamic limit, keeping only the leading terms will give a good approximation. Thus we can neglect the last term in the expression under the partial derivative with respect to $N_f$ once

$N/N^{ini}$ has become sufficiently small. We are left with the Fokker-Planck equation of the same type as the one already studied in [13], but with a different coefficient.

$$-\frac{\partial P}{\partial N}=\frac{\partial}{\partial N_f}\left(\frac{N_f}{N}+\frac{\mu N}{N^{ini}}\right)P+\frac{1}{2}\frac{\partial^2 P}{\partial N_f^2}, \tag{8}$$

where $\mu=(1-2\omega_1)/(1-\omega_1)$.

We introduce the variables

$$x=\frac{N_f}{\sqrt{N}} \text{ and } y=\frac{N}{(N^{ini}/\mu)^{2/3}} \tag{9}$$

and the function $f(x,y)=(N^{ini}/\mu)^{\gamma}P(N_f,N)$. The free parameter $\gamma$ will be fixed below by the condition that the probability distribution of the number of nonfrozen nodes is normalized. The Fokker-Planck equation then becomes

$$y\frac{\partial f}{\partial y}+f+\left(\frac{x}{2}+y^{3/2}\right)\frac{\partial f}{\partial x}+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}=0. \tag{10}$$

Let $W(N)$ denote the probability that $N$ nodes are left at the moment where $N_f$ reaches the value zero. It is

$$W(N)=\int_0^\infty P(N_f,N)dN_f-\int_0^\infty P(N_f,N-1)dN_f.$$

Consequently,

$$W(N)=\frac{\partial}{\partial N}\int_0^\infty P(N_f,N)dN_f=(N^{ini}/\mu)^{-\gamma-1/3}\frac{\partial}{\partial y}\sqrt{y}\int_0^\infty f(x,y)dx$$
$$\equiv(N^{ini}/\mu)^{-\gamma-1/3}G(y)$$

with a scaling function $G(y)$. $W(N)$ must be a normalized function,

$$\int_0^\infty W(N)dN=(N^{ini}/\mu)^{-\gamma-1/3+2/3}\int_0^\infty G(y)dy=1.$$

This gives $\gamma=1/3$. This condition is independent of the parameters of the model, and therefore $G(y)$ and $f(x,y)$ are independent of them too. Now, we have

$$W(N)=(N^{ini}/\mu)^{-2/3}G(y).$$

The mean number of nonfrozen nodes is

$$\bar{N}=\int_0^\infty NW(N)dN=(N^{ini}/\mu)^{2/3}\int_0^\infty G(y)ydy,$$

which is proportional to $(N^{ini})^{2/3}$.

The probability $W_2(N_2)$ that $N_2$ nodes are left in container $\mathcal{N}_2$ at the moment where container $\mathcal{F}$ becomes empty is obtained from the relation

$$N_2=\frac{1}{1-\omega_1}\frac{N^2}{N^{ini}}-\frac{1}{1-\omega_1}\frac{N^3}{(N^{ini})^2}.$$

Since $W(N)dN=W_2(N_2)dN_2$, we find that the mean number of nodes left in container $\mathcal{N}_2$ is

$$\bar{N}_2=\int_0^\infty N_2W_2(N_2)dN_2$$
$$=\int_0^\infty N_2W(N)dN$$
$$=\frac{1}{(\mu)^{1/3}(1-2\omega_1)}(N^{ini})^{1/3}\int_0^\infty y^2G(y)dy+\frac{1}{\mu}\int_0^\infty y^2G(y)dy$$
$$\sim(N^{ini})^{1/3}.$$

In the same manner we find for the number of nodes left in container $\mathcal{N}_3$

$$\bar{N}_3=\int_0^\infty N_3W_3(N_3)dN_3$$
$$=\int_0^\infty N_3W(N)dN$$
$$=\frac{\beta(1-2\omega_1)^2}{(1-\omega_1)^2}\int_0^\infty y^3G(y)dy\sim\text{const}.$$

Thus we have shown that the number of nonfrozen nodes scales with network size $N^{ini}$ as $(N^{ini})^{2/3}$, with most of these nodes receiving only one input from other nonfrozen nodes. The number of nonfrozen nodes with two nonfrozen inputs scales as $(N^{ini})^{1/3}$ and the number of nodes with three such inputs is independent of the network size.

## V. SPECIAL POINTS AND CANALYZING FUNCTIONS

For $\omega_1=1/2$, the second term in the Langevin equation (6) is zero. In this case the next order term has to be taken into account since it is the leading one now. We will see that the mechanism of creating the frozen core is different for such systems, but in the end we will find the same scaling behavior of the number of nonfrozen nodes.

Now we have to consider the modified Langevin equation

$$\frac{dN_f}{dN}=\frac{N_f}{N}+2(1-\beta)\left(\frac{N}{N^{ini}}\right)^2+\xi \tag{11}$$

and the corresponding Fokker-Planck equation

$$-\frac{\partial P}{\partial N}=\frac{\partial}{\partial N_f}\left[\frac{N_f}{N}+2(1-\beta)\left(\frac{N}{N^{ini}}\right)^2\right]P+\frac{1}{2}\frac{\partial^2 P}{\partial N_f^2}. \tag{12}$$

We again introduce new variables

$$x=\frac{N_f}{\sqrt{N}} \text{ and } y=\left(\frac{(N^{ini})^2}{2(1-\beta)}\right)^{-4/5}N^2 \tag{13}$$

and the function $f(x,y)=\left(\frac{(N^{ini})^2}{2(1-\beta)}\right)^{\gamma}P(N_f,N)$. The Fokker-Planck equation then becomes

$$2y\frac{\partial f}{\partial y}+f+\left(\frac{x}{2}+y^{5/4}\right)\frac{\partial f}{\partial x}+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}=0.$$

For the probability that $N$ nodes are left when $N_f$ reaches zero we obtain

$$W(N) = \left( \frac{(N^{ini})^2}{2(1-\beta)} \right)^{-2/5} \tilde{G}(y)$$

with a new scaling function $\tilde{G}$. We have used the fact that this probability has to be normalized, which gives $\gamma = 1/5$.

Using this result, we find for the mean number of nonfrozen nodes

$$\bar{N} = \int_0^\infty NW(N)dN = \frac{1}{2}\left( \frac{(N^{ini})^2}{2(1-\beta)} \right)^{2/5} \int_0^\infty \tilde{G}(y)dy \sim (N^{ini})^{4/5}. \tag{14}$$

For the mean number of nonfrozen nodes left in containers $\mathcal{N}_2$ and $\mathcal{N}_3$ we find

$$\begin{aligned} \bar{N}_2 &= \int_0^\infty N_2 W_2(N_2) dN_2 \\ &= \int_0^\infty N_2 W(N) dN \\ &= \frac{(N^{ini})^{3/5}}{[2(1-\beta)]^{4/5}} \int_0^\infty y^{1/2} \tilde{G}(y)dy - \frac{(N^{ini})^{2/5}}{[2(1-\beta)]^{6/5}} \int_0^\infty y\tilde{G}(y)dy \\ &\sim (N^{ini})^{3/5} \end{aligned} \tag{15}$$

and

$$\begin{aligned} \bar{N}_3 &= \int_0^\infty N_3 W_3(N_3) dN_3 \\ &= \int_0^\infty N_3 W(N) dN \\ &= \frac{\beta}{2} \frac{(N^{ini})^{2/5}}{[2(1-\beta)]^{6/5}} \int_0^\infty y\tilde{G}(y)dy \sim (N^{ini})^{2/5}. \end{aligned} \tag{16}$$

We see that the number of nodes which become frozen due to the influence of the constant functions is smaller than in the case of other critical networks. When we look at the parameters for these networks more closely, we see that these networks are effectively canalyzing with two inputs per node. The probability that a node with two inputs is going to freeze during one time step is $\omega_1 = 1/2$ and this means that the network has Boolean functions such that nodes with two nonfrozen inputs effectively belong to the $\mathcal{C}_1$ or $\mathcal{C}_2$ class of Boolean functions with two variables, i.e., canalyzing functions. The class $\mathcal{C}_1$ contains those functions that depend only on one of the two variables, but not on the other one. The class $\mathcal{C}_2$ contains the remaining canalyzing functions, where one state of each input fixes the output. It has been shown in [14] that in $K=2$ networks with only this type of functions another mechanism of creating the frozen core arises. The only condition for this is that the number of nodes from class $\mathcal{C}_2$ is large enough. We will show that it is exactly what happens in the networks we are analyzing now. The number of nonfrozen nodes with two inputs and canalyzing $\mathcal{C}_2$ functions is here large enough to allow for the creation of the self-freezing loops that are going to increase the number of frozen nodes and thus change the scaling of the nonfrozen nodes from $(N^{ini})^{4/5}$ to $(N^{ini})^{2/3}$.

## VI. CREATING SELF-FREEZING LOOPS AND THEIR EFFECT

We are now considering a reduced network consisting of those nodes that are not frozen through the influence of the nodes with constant functions. The size of this network is $N \simeq (N^{ini})^{4/5}$, most of the nodes have one nonfrozen input, $N_2 \simeq (N^{ini})^{3/5}$ have two, and $N_3 \simeq (N^{ini})^{2/5}$ have three nonfrozen inputs. Nodes with two nonfrozen inputs have a probability to freeze $\omega_1 = 1/2$ and as such effectively have canalyzing Boolean functions of two arguments, belonging to $\mathcal{C}_1$ or $\mathcal{C}_2$ class. So, the number of nodes with two nonfrozen inputs that belong to the $\mathcal{C}_2$ class has to be $\simeq (N^{ini})^{3/5}$ as it is the fraction of all nonfrozen nodes with two inputs.

Let us now assume that there exist groups of nodes that fix each other's value and do not respond to changes in nodes outside this group. The simplest example of such a group is a loop of $\mathcal{C}_2$ nodes where each node canalyzes (fixes) the state of its successor once it settles on its majority bit (the one occurring three times in its update function table). These loops, introduced in [14], are called *self-freezing loops*. They can also contain chains of nodes with one nonfrozen input or with two nonfrozen inputs and a $\mathcal{C}_1$ function between $\mathcal{C}_2$ nodes. If a chain between two $\mathcal{C}_2$ nodes as a whole inverts the state of the first $\mathcal{C}_2$ node, the inverted majority bit of the first $\mathcal{C}_2$ node has to canalyze the second $\mathcal{C}_2$ node. The only effect of nodes with $\mathcal{C}_1$ functions and those with one nonfrozen input in such loops is to delay the signal propagation between two adjacent $\mathcal{C}_2$ nodes. The procedure of finding self-freezing loops is explained in detail in [14]. The number of nodes on self-freezing loops is there found by mapping the problem of finding a self-freezing loop in a $\mathcal{C}_2$ network onto the problem of finding the relevant nodes sitting on relevant loops in a critical network that contains no canalyzing functions at all, but only reversible (where the output is changed whenever one of the inputs is changed) and constant functions. Using results for these reversible networks obtained in [13] it was found that the number of nodes on self-freezing loops scales as $\sim N^{1/3}$ where $N$ is the number of $\mathcal{C}_2$ nodes.

Obviously, nodes depending on or canalyzed by the frozen nodes of the self-freezing loops freeze also, and such nodes may lead to the freezing of further nodes, etc. We can introduce a dynamical process in order to determine the total number of nodes that become frozen because of the self-freezing loops. This process is almost the same as the one we have used for identifying the influence of the constant functions on the networks dynamics. We again have four containers where the nodes left after determining the influence of the nodes with constant functions are placed. Initially nodes found to be on the self-freezing loops are going to be moved from the container with nodes with two inputs, $\mathcal{N}_2$, to the container $\mathcal{F}$. Thus the initial number of nodes in the containers is going to be $N_f^0 = [(N^{ini})^{3/5}]^{1/3} = (N^{ini})^{1/5}$, $N_2^0 = (N^{ini})^{3/5} - N_f^0 \simeq (N^{ini})^{3/5}$ and $N_3^0 = (N^{ini})^{2/5}$, and the total number of nodes is $N^0 = (N^{ini})^{4/5}$. Now we run the same dynamical pro-

cess as before determining influence of the nodes from the frozen loops on the rest of this reduced network one by one and then removing them from the system. At the end of this process we will again have nodes in the container $\mathcal{N}_2$. They can now make new self-freezing loops made of $\mathcal{C}_2$ nodes with the chains of nodes with one nonfrozen input between them. We can then again move $N_2^{1/3}$ nodes that are on the new self-freezing loops to the container $\mathcal{F}$ and run the same process again. We can even take over the values of $N_1$, $N_2$, and $N_3$ and $N$ at the end of the first process, since $N_2{}^{1/3}$ frozen nodes moved from container $\mathcal{N}_2$ are negligible in comparison to $N_2$. These processes can be repeated as long as the number of nodes of type $\mathcal{C}_2$ is large enough to allow for the creation of self-freezing loops. The equations for the change of $N_3$ and $N_2$ nodes

$$\Delta N_3 = -\frac{3N_3}{N},$$

$$\Delta N_2 = -\frac{2N_2}{N} + \frac{2}{\beta}\frac{N_3}{N} \tag{17}$$

apply together to all the successive processes of freezing the network through the influence of nodes of the self-freezing loops. Between each two of them the new self-freezing loops have been found and moved from the container with $N_2$ nodes allowing for the new process to start. The equation for $N$ is $\Delta N=-1$, as before. The solution of these equations is obtained by going to differential equations for $dN_2/dN$ and $dN_3/dN$. Using the values of $N$, $N_2$, and $N_3$, found in Eqs. (14)–(16), as initial values of the variables, these differential equations have the solution

$$N_3 = \frac{N_3^0}{(N^0)^3}N^3, \tag{18}$$

$$N_2 = \frac{N_2^0 + (2/\beta)N_3^0}{(N^0)^2}N^2 - \frac{2N_3^0}{\beta(N^0)^3}N^3. \tag{19}$$

The number of remaining $\mathcal{N}_1$ nodes increases in the second process, the number of $\mathcal{C}_2$ (those in container $\mathcal{N}_2$) nodes decreases, thus leading to an increasing weight of $\mathcal{N}_1$ nodes in the nonfrozen network.

The repeated process of identifying generalized self-freezing loops and the nodes frozen by them breaks down when the remaining nonfrozen nodes cannot be considered as an effective $\mathcal{C}_2$ network anymore. This happens when in the process of creating self-freezing loops the probability that a $\mathcal{C}_2$ node is going to be attached to the end of the chain of nodes with one nonfrozen input (thus making closing self-freezing loop possible) becomes of the same order of magnitude as the probability that this chain becomes a loop. Since the mean size of the loops of nodes with one input is found to be of the order of $\sqrt{N}$ [15] the assembly of the self-freezing loop becomes improbable when $N_2 \sim \sqrt{N}$.

This condition gives to leading order

$$\frac{(N^0)^2}{N_2^0} \sim N^{3/2} \tag{20}$$

or $N \sim (N^{ini})^{2/3}$. We again have the same scaling of the number of nonfrozen nodes with the network size. The scaling of the number of nonfrozen nodes with two and three nonfrozen inputs with the network size we find from Eqs. (19) and (18) to be $N_2 \sim (N^{ini})^{1/3}$ and $N_3 \sim$ const. This is the same scaling we have for the case of all other critical networks investigated until now.

When finding the number of nodes on the self-freezing loops and defining our second process we assumed that there the influence of the nodes with three nonfrozen inputs per node is negligible. We can check if our assumption was justified. In the beginning of this process the number of nodes with three inputs was $N_3^0 \simeq (N^{ini})^{2/5}$. The number of nodes that are initially on self-freezing loops is $(N_2^0)^{1/3} = (N^{ini})^{1/5}$. The mean number of nodes with three inputs on the self-freezing loops is then

$$(N_2^0)^{1/3}\frac{N_3^0}{N_2^0} = \text{const}.$$

In the limit of large network size, only a few (if any) self-freezing loops are destroyed by nodes with three nonfrozen inputs, and this does not change the scaling behavior of the number of nodes on self-freezing loops.

## VII. NETWORKS WITHOUT CONSTANT FUNCTIONS

### A. Case $\omega_1=1/2$, $\omega_2=1/3$

Until now, we have assumed that the network has nodes with constant functions. In this section, we consider networks without constant functions, i.e., with $\beta=1$. The criticality condition (3) then becomes

$$3(1-\omega_1)(1-\omega_2) = 1.$$

Although the criticality condition was derived under the assumption that the network has a nonvanishing proportion of frozen nodes (i.e., that $\beta<1$), it can be extended to $\beta=1$, since it is valid for any $\beta$ arbitrarily close to 1. Furthermore, decreasing $\beta$ slightly for fixed $\omega_1$ and $\omega_2$ moves the system to the frozen phase, indicating that a system satisfying the criticality condition with $\beta=1$ is at the boundary of the frozen phase. As we will see, the value of the parameters in the critical networks without constant functions we are considering here is allowing the formation of the self-freezing loops and leads to the frozen core of the same size as for all the other critical networks. Canalyzing networks and threshold networks are examples of this category of networks, and they are considered important for biological applications.

The procedure of creating self-freezing loops in the case of networks with nodes with two nonfrozen inputs was introduced and explained in detail in [14]. It is the same procedure we have used in the previous section. Using a similar line of arguments we can explain the assembly of the self-freezing loops for the networks with three inputs per node determined with parameters being $\omega_1=1/2$, $\omega_2=1/3$, and $\beta=1$. In this case there is a mapping of the problem of finding

the nodes on the self-freezing loops in this network onto the problem of finding the relevant nodes on relevant loops in critical network with three inputs per node and only reversible and constant functions, i.e., with $\omega_1=\omega_2=0$ and $\beta=1/3$. Self-freezing loops are found by starting with a node and keeping track of the connection to those inputs that are able to canalyze this node if they are canalyzed themselves. This procedure is iterated for these input nodes, etc., until a loop is formed or until it has to stop because no canalyzing inputs are found. Similarly, relevant loops in a critical network with $\omega_1=\omega_2=0$ are found by starting with a node and keeping track of the connection to those inputs that do not have a constant function. This procedure is iterated for the nonfrozen inputs, etc., until a loop is formed or until it has to stop because no nonfrozen inputs are found. In both cases, a connection to an input is made with probability 1/3, showing that the two processes can be mapped on each other. As we will show in Sec. IX below, in critical networks with three inputs per node and nonzero fraction of frozen nodes the number of relevant nodes on relevant loops scales as $(N^{ini})^{1/3}$. Therefore we conclude that in the network with $\omega_1=\omega_2=0$, the number of nodes on self-freezing loops scales also as $(N^{ini})^{1/3}$.

We can now proceed just as in the previous section, but with $\beta=1$ and $N_j^0=N_j^{ini}$. We continue making self-freezing loops and determining which nodes are frozen by them until $N_2\sim\sqrt{N}$. Inserting this condition in Eq. (19), we find to leading order

$$2\frac{N^{3/2}}{N^{ini}}=1,$$

leading again to $N\sim(N^{ini})^{2/3}$.

### B. General case

Now, let us turn to the case $\beta=1$ with $\omega_1<1/2$. (The situation $\omega_1>1/2$ is not possible for nonfrozen Boolean functions with two inputs.) The probability that a node we do not know anything about freezes when connected to a frozen node is now $\omega_2>1/3$. Every node has three inputs and this frozen node could be any of them. This means that on an average a node can be frozen by more than one input, and the self-freezing components we look for in the network here consist of at least as many nodes as those in the previous section. However, we do not need to know the exact number of frozen nodes in these components. We will build only one self-freezing loop and move its $(N^{ini})^{1/3}$ nodes to the container $\mathcal{F}$. Then we start the calculation of Sec. II by setting $\beta=1-(N^{ini})^{-2/3}$. Since $\omega_1<1/2$, the leading-order terms of the calculation performed in Sec. II are retained in this case, and we can take over all the main results of that section. In particular, it follows that a single self-freezing loop is sufficient to generate the entire frozen core, and we do not need to identify other self-freezing loops. As before, the number of nonfrozen nodes scales as $(N^{ini})^{2/3}$.

## VIII. GENERALIZATION TO LARGER $K$

The process introduced in Sec. II can easily be generalized to networks with $K>3$. We first consider again the case $\beta<1$. For a network with $K$ inputs we define a set of parameters $\beta$ and $\omega_i$ with $i\in[1,K-1]$. $\beta$ is again a fraction of the nonfrozen nodes and $\omega_i$ is the probability that a nonfrozen node that has $K-i$ inputs from frozen nodes freezes when receiving another frozen input in our process. These $K$ parameters are going to define completely the class of networks we observe in the process. Using the deterministic description of the process analogous to the one described in Sec. III we find the criticality condition for networks with any $K$:

$$K(1-\omega_1)(1-\omega_2)\cdots(1-\omega_{K-1})=1. \tag{21}$$

Introduction of noise in the process gives the Langevin equation

$$\frac{dN_f}{dN}=\frac{N_f}{N}+\sum_{i=1}^{K-1}f_i(\omega_1,\ldots,\omega_i)\left(\frac{N}{N^{ini}}\right)^i+\xi, \tag{22}$$

where the $f_i(\omega_1,\ldots,\omega_i)$ are functions of the parameters of the system obtained from the stochastic process They satisfy $f_i(\omega_1,\ldots,\omega_i)=0$ when $\omega_j=1/(j+1)$ for all $j\in[1,i]$. We see that in this general Langevin equation the leading term in $N$ is the same as in Eq. (6). Therefore we find that in the thermodynamic limit the number of nonfrozen nodes scales in critical networks as $(N^{ini})^{2/3}$ with the network size.

Just like in the $K=3$ networks, parameter values can be such that one or more of the leading terms in the Langevin equation vanish. These special points in the parameter space describe networks where the Boolean functions are such that the nodes left nonfrozen after determining the influence of the frozen nodes in our process can additionally generate self-freezing loops. Their influence on the rest of the network has to be determined by generalization of the process introduced in Sec. VI. The number of classes of special points will increase with $K$, leading to a hierarchy of special points. For each $K$, there are $K-3$ classes of points in parameter space that are equivalent to the special points of networks with $K-1$ inputs per node (that is they have the same leading term in the Langevin equation), and one new class of special points where only the last term in the Langevin equation (22) is nonzero. Furthermore, there is the case $\beta=1$. As an illustration, in the case $K=4$ there are two classes of special points for $\beta<1$. One of them has $\omega_1=1/2$. In this case, the influence of the frozen nodes will lead to $(N^{ini})^{4/5}$ nonfrozen nodes. Boolean functions of the nodes with two nonfrozen inputs and the number of them left after the first process are such that self-freezing loops are made and their influence will again give $(N^{ini})^{2/3}$ as the number of nonfrozen nodes in the network. This case can obviously be reduced to the $K=3$ network. The other class of special points is obtained when the parameters of the network are $\omega_1=1/2$ and $\omega_2=1/3$. In this case, $(N^{ini})^{6/7}$ nodes will be left nonfrozen after determining the influence of the frozen nodes. One can easily show that the creation of self-freezing loops is possible and that their influence leads to a number of nonfrozen nodes that scales as $(N^{ini})^{2/3}$ with the network size.

For general values of $K$, the $K-2$ classes of special points with $\beta<1$ are given by the condition $\omega_j=1/(j+1)$ for all $j\in[1,i]$ where $i$ takes for every class one of the values from the interval $[1,\ K-2]$. This means that $f_1=0,\ldots,f_i=0$

in the Langevin equation (22) and the term $f_{i+1}(\omega_1,\ldots,\omega_{i+1})(N/N^{ini})^{i+1}$ is the leading one. The nodes left nonfrozen after determining the influence of the nodes with constant functions scale with the network size as $(N^{ini})^{(2i+2)/(2i+3)}$. The numbers and Boolean functions of the nodes with $k\in[2,i+1]$ nonfrozen inputs are such that they allow for the creation of the self-freezing loops, and their influence will for each of these special points, i.e., for each $i\in[1,K-2]$, reduce the number of nonfrozen nodes to $(N^{ini})^{2/3}$.

For networks without constant functions (that is with $\beta=1$) the frozen core arises only because of the creation of self-freezing loops and their effect on the network. Just like for all other parameter values, there is straightforward generalization of the analysis performed for this type of network in the case when $K=3$ in Sec. VII. In the case when $\omega_i=1/(i+1)$ for all $i\in[1,K-1]$ there exists again a mapping of the self-freezing loops on the relevant loops of a $K$ critical network with only reversible and nonfrozen functions, from which it follows that the number of nodes that are initially on self-freezing loops scales as $(N^{ini})^{1/3}$. The process described in Sec. VI can then be generalized to these networks. For any other choice of parameters satisfying the criticality condition (21) for $\beta=1$, self-freezing loops can also be formed, and after moving only one of them in the container with frozen nodes we will have the same process as for the one of the classes of critical networks with $\beta<1$ that were already studied. Scaling of the number of nonfrozen nodes in the critical networks without frozen nodes and any fixed number of inputs will be the same as in all other critical networks.

Let us end this section by noting that there is another class of special points when the Boolean functions are chosen such that each of them responds only to one of the $K$ inputs. In this case, the network is effectively a $K=1$ network, since for each node those $K-1$ inputs to which the node does not respond, can be cut off. In the calculations of the previous sections we have always assumed that a nonvanishing proportion of functions is not of this type.

## IX. RELEVANT NODES AND THE NUMBER AND LENGTH OF ATTRACTORS

Relevant nodes are the nodes whose state is not constant and that control at least one relevant node. These nodes determine completely the number and period of attractors. In the previous sections, we have shown that the number of nonfrozen nodes scales as $(N^{ini})^{2/3}$ for any critical network. We have also seen that among them there are only $(N^{ini})^{1/3}$ nodes having two nonfrozen inputs, and that the number of nonfrozen nodes with more than two nonfrozen inputs vanishes in the thermodynamic limit. The nonfrozen nodes can now be connected to a network. This is a reduced network, where all frozen nodes have been cut off. In [13], we defined a stochastic process for the formation of this reduced network and the identification of the relevant nodes for critical $K=2$ networks. The relevant nodes are determined by removing iteratively nodes that are not relevant because they influence only frozen and irrelevant nodes. The number of relevant nodes was found to scale as $(N^{ini})^{1/3}$, and the scaling function characterizing their probability distribution depends on the parameters of the model.

The scaling of the number of nonfrozen nodes as well as the scaling of the number of nonfrozen nodes with two nonfrozen inputs as a function of the network size is the same for every critical network, as we have shown in this paper. Since the fraction of nodes with more than two nonfrozen inputs is vanishing in the thermodynamic limit, the network of nonfrozen nodes, which is the starting point for the process of determining the relevant nodes, is the same as in the $K=2$ case. So, we can conclude that the results for the scaling of the number of relevant nodes found in [13] for the $K=2$ critical networks are valid for any critical network. The number of relevant nodes in critical networks scales as $(N^{ini})^{1/3}$ with the network size. Among them are a constant number of relevant nodes with two relevant inputs and a vanishing number of relevant nodes with more than two relevant inputs in the limit $N^{ini}\to\infty$. If only these nodes and the links between them are considered, they form loops with possibly additional links and chains of relevant nodes within and between loops.

It follows that all critical networks with $K>1$ show the same scaling behavior. The only exception is the case $K=1$, which is different because there is no frozen core.

As we have shown in [13], we can derive properties of attractors from the results for the relevant nodes. In particular, we can take over the result of [13] that all relevant components apart from a finite number are simple loops, and that the mean number and length of attractors increases faster than any power law with the network size.

## X. CONCLUSIONS

In this paper, we have considered the limit of large network size, and we have found the scaling behavior of the number of nonfrozen nodes, of the number of nonfrozen nodes with more than one nonfrozen input, of the number of relevant nodes, and of the number of relevant nodes with more than one relevant input in a general class of critical random Boolean networks with fixed number of inputs per node. The mean values of these quantities scale with network size $N^{ini}$ as a power law in $N^{ini}$, with the exponents being the same for any critical network. No matter what the distribution of the Boolean functions is and how many inputs per node the critical network has, the number of nonfrozen nodes scales with the network size as $(N^{ini})^{2/3}$, the number of nonfrozen nodes with two nonfrozen inputs scales as $(N^{ini})^{1/3}$, the exponent for the number of nonfrozen nodes with three nonfrozen inputs is zero, and it is $-n/3$ for the number of nonfrozen nodes with $n+3$ nonfrozen inputs. The number of relevant nodes scales always as $(N^{ini})^{1/3}$, with a constant number of them having two inputs and a vanishing proportion having more than two.

It follows that all critical random Boolean networks with $K>1$ belong to the same class of systems. Changing the weights of the different Boolean functions (for instance, by choosing threshold networks or canalyzing networks) or changing the number of inputs per node (which might make the model more relevant for biological applications) will not

change the scaling of the number of nonfrozen and relevant nodes with the size of the network, and it will not change the fact that the number and length of attractors increases faster than any power law with the network size, as long as the network is critical. Using a different method, Samuelsson and Socolar have recently also found that the number of nonfrozen nodes scales in the same way for all $K>1$ critical networks [16].

From the calculations performed in this paper it can be concluded that the results are also valid for networks that have nodes with different values of $K$. If $K_{max}$ is the largest number of inputs occuring in the network, we can set $K=K_{max}$, and we can view nodes with less inputs as nodes with $K_{max}$ inputs, but with a function that does not depend on all of its inputs. In contrast, our results cannot be generalized to networks with a broad distribution of the number of outputs. The method employed in this paper is based on a Poissonian distribution of the number of outputs, and is most likely valid also for other distributions as long as the second moment of the number of outputs is finite. This can for instance be concluded from the analogy between the propagation of activity in a Boolean network and percolation on a directed graph, for which many results are known [17].

The finding that the number and length of attractors in critical Boolean networks increases superpolynomially with network size is detrimental to the hypothesis that these networks are models of gene regulation networks, where only a limited number of dynamic pathways should exist. However, by considering asynchronous update instead of parallel update and by requiring that dynamics should be robust with respect to fluctuations in the update sequence, the number of attractors reduces to a power law in system size, which is more realistic than the superpolynomial growth [18,19]. The method presented in this paper is independent of the updating scheme, and the scaling of the number of nonfrozen and relevant nodes is therefore the same for asynchronous update as for parallel update. The relevant components are consequently also the same. With the insights obtained in the present paper, we can immediately apply the results for asynchronous update in $K=2$ critical networks to critical networks with larger values of $K$, and we can conclude that the number of attractors in critical networks with asynchronous update increases as a power law of the system size.

Finally, let us consider networks where the connections between nodes are not made at random, but that show some degree of clustering. Such networks have a finite proportion of nodes that have correlated inputs and that can therefore become frozen, e.g., because their inputs are always in the same state. In contrast, the randomly wired networks considered in the present paper have only a limited and small number of nodes with correlated inputs even in the thermodynamic limit of infinite network size. For small-world networks, which have a high degree of clustering, our method for determining the frozen core is not valid because it is based on the assumption that nodes choose their inputs independently from each other. Small-world networks need therefore a separate analytical treatment, which has not been done so far.

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[1] S. Kauffman, C. Peterson, B. Samuelsson, and C. Troein, Proc. Natl. Acad. Sci. U.S.A. **100**, 14796 (2003).
[2] S. A. Kauffman, J. Theor. Biol. **22**, 437 (1969).
[3] B. Derrida and Y. Pomeau, Europhys. Lett. **1**, 45 (1986).
[4] B. Derrida and D. Stauffer, Europhys. Lett. **2**, 739 (1986).
[5] M. Aldana-Gonzalez, S. Coppersmith, and L. P. Kadanoff, in *Perspectives and Problems in Nonlinear Science, A Celebratory Volume in Honor of Lawrence Sirovich*, edited by E. Kaplan, J. E. Mardsen, and K. R. Sreenivasan, Springer Applied Mathematical Sciences Series (Springer-Verlag, New York, 2003), pp. 23–89.
[6] H. Flyvbjerg, J. Phys. A **21**, L955 (1988).
[7] H. Flyvbjerg and N. J. Kjær, J. Phys. A **21**, 1695 (1988).
[8] U. Bastolla and G. Parisi, Physica D **115**, 203 (1998).
[9] U. Bastolla and G. Parisi, Physica D **115**, 219 (1998).
[10] J. E. S. Socolar and S. A. Kauffman, Phys. Rev. Lett. **90**, 068702 (2003).
[11] B. Samuelsson and C. Troein, Phys. Rev. Lett. **90**, 098701 (2003).
[12] B. Drossel, Phys. Rev. E **72**, 016110 (2005).
[13] V. Kaufman, T. Mihaljev, and B. Drossel, Phys. Rev. E **72**, 046124 (2005).
[14] U. Paul, V. Kaufman, and B. Drossel, Phys. Rev. E **73**, 026118 (2006).
[15] B. Drossel, T. Mihaljev, and F. Greil, Phys. Rev. Lett. **94**, 088701 (2005).
[16] B. Samuelsson and J. E. Socolar e-print nlin.CG/0605047.
[17] N. Schwartz, R. Cohen, D. ben-Avraham, A.-L. Barabasi, and S. Havlin, Phys. Rev. E **66**, 015104(R) (2002).
[18] K. Klemm and S. Bornholdt, Phys. Rev. E **72**, 055101(R) (2005).
[19] F. Greil and B. Drossel, Phys. Rev. Lett. **95**, 048701 (2005).